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On the steady laminar flow of an incompressible viscous fluid in a curved pipe of elliptical cross-section

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A literature survey (Berger, Talbot & Yao 1983) indicates that laminar viscous flow in curved pipes has been extensively investigated. Most of the existing analytical results deal with the case of circular cross-section. The important studies dealing with elliptical cross-sections are mainly due to Thomas & Walters (1965) and Srivastava (1980). The analysis of Thomas & Walters is based on Dean's (1927, 1928) approach in which the simplified forms of the momentum and continuity equations have been used. The analysis of Srivastava is essentially a seminumerical approach, in which no explicit expressions have been presented.

In this paper, using elliptic coordinates and following the unsimplified formulation of Topakoglu (1967), the flow in a curved pipe of elliptical cross-section is analysed. Two different geometries have been considered: (i) with the major axis of the ellipse placed in the direction of the radius of curvature; and (ii) with the minor axis of the ellipse placed in the direction of the radius of curvature. For both cases explicit expressions for the first term of the expansion of the secondary-flow stream function as a function of the ellipticity ratio of the elliptic section have been obtained. After selecting a typical numerical value for the ellipticity ratio, the secondary-flow streamlines are plotted. The results are compared with that of Thomas & Walters. The remaining terms of the expansion of the flow field are not included, but they will be analysed in a future paper.

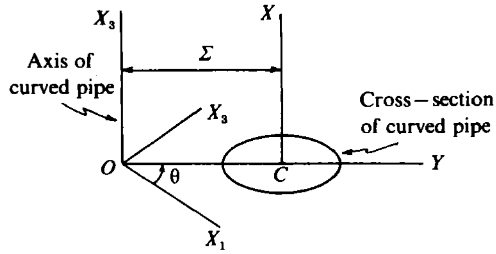
1. Introduction

In determining the flow field in a curved pipe with elliptic periphery it seems natural to use elliptic coordinates. In addition to this any available velocity field expressed in elliptic coordinates facilitates further investigations of problems involving flows in curved elliptical pipes. Elliptic coordinates are thus selected to represent the orthogonal coordinates in the cross-sectional plane of the curved pipe. The selection of coordinate axes and other pertinent quantities are shown in figures 1 and 2, where the axis OX_3 represents the axis of symmetry for the curved pipe. The origin of axes CX and CY in the cross-sectional plane is taken at the point C.

The semimajor and the semiminor axes of the elliptic periphery are denoted by A_1 and B_1 respectively. The radius of curvature of the centreline of the curved pipe is denoted by Σ. The corresponding dimensionless radius of curvature, using the horizontal semiaxis of the section as unit length, is defined as

$$\sigma = \Sigma/A, \tag{1.1}$$

where A is the horizontal semiaxis of the section ($A = A_1$ for a horizontally placed ellipse, $A = B_1$ for a vertically placed ellipse).



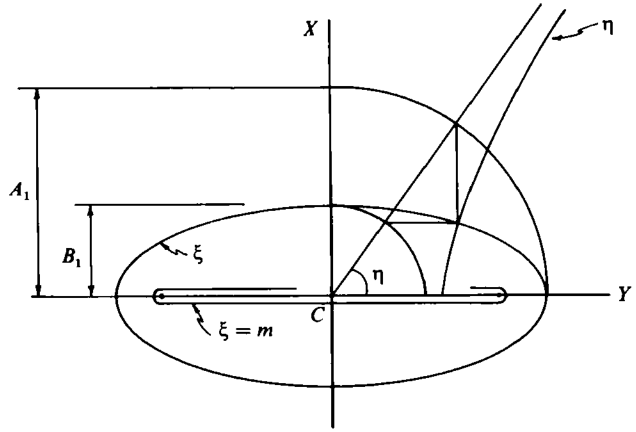


FIGURE 1. Definition of coordinates for horizontally placed ellipse.

The ellipticity of the periphery is defined by the factor

$$m=\left[\left(1-\frac{B_1}{A_1}\right)\Big/\left(1+\frac{B_1}{A_1}\right)\right]^{\frac{1}{2}}. \tag{1.2}$$

The dimensionless variables are defined by

$$\left.\begin{aligned} X &= Ax, \quad Y = Ay, \\ \Psi &= A\nu\psi, \quad W = \frac{\nu}{A}w, \\ K &= \frac{\nu^2}{A^2}k\left(=\frac{1}{\rho}\frac{\partial p}{\partial \theta}\right), \end{aligned}\right\} \tag{1.3}$$

where (X, Y) and (x, y) are respectively dimensional and dimensionless coordinates along the X- and Y-axes, W and w are the dimensional and dimensionless velocity components in the θ-direction, Ψ and ψ are the dimensional and dimensionless stream functions in the cross-sectional plane, and ν, ρ and p are the kinematical viscosity, density and the pressure.

The velocity components along the X- and Y-axes are related to the stream function ψ by

$$U=\frac{1}{Y}\Psi_Y, \quad V=-\frac{1}{Y}\Psi_X. \tag{1.4}$$

It is convenient to introduce a new stream function ϕ, as

$$\phi=\frac{1}{\sigma}\psi, \tag{1.5}$$

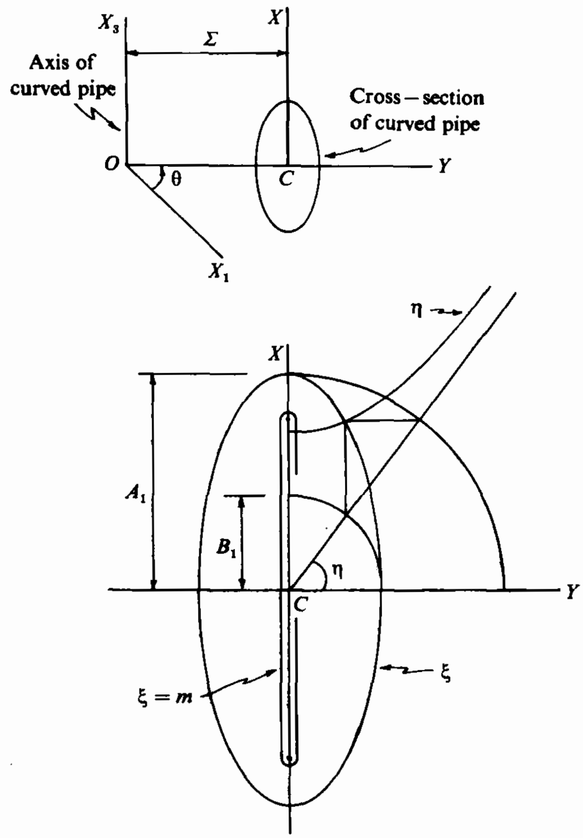


FIGURE 2. Definition of coordinates for vertically placed ellipse.

and a constant k_0 by the relation

$$k = -\sigma k_0. \tag{1.6}$$

The constant k_0 is related to the Reynolds number of the flow by

$$k_0 = 4Re, \tag{1.7}$$

provided that the Reynolds number for the flow in the curved pipe is defined as the Reynolds number for a flow in a straight circular pipe of radius A and maintained under the same pressure gradient as that of the curved pipe. The pressure gradient in the curved pipe is measured along the centreline circle.

The dimensionless primary-flow velocity component w and the dimensionless stream function ϕ satisfy the following equations (cf. Topakoglu 1967):

$$\left.\begin{aligned} \nabla^2 w &= \frac{1}{(y+\sigma)^2}\, w(1-\sigma\phi_x) + \frac{1}{y+\sigma}\left[\sigma\,\frac{\partial(\phi, w)}{\partial(y,x)} - w_y - \sigma k_0\right], \\ \nabla^4\phi &= \frac{1}{\sigma}\,(w^2)_x + \frac{1}{y+\sigma}\left[2\nabla^2\phi_y + \sigma\,\frac{\partial(\phi, \nabla^2\phi)}{\partial(y,x)}\right] \\ &\quad - \frac{1}{(y+\sigma)^2}\left[3\phi_{yy} + 3\,\frac{\partial(\phi,\phi_y)}{\partial(y,x)} - 2\sigma\phi_x\nabla^2\phi\right] + 3\,\frac{1}{(y+\sigma)^3}\,\phi_y(1-\sigma\phi_x), \end{aligned}\right\} \tag{1.8}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}.$$

2. Expansions of primary and secondary flows in terms of curvature

The flow field can be expanded in terms of the curvature

$$\lambda = 1/\sigma \tag{2.1}$$

into power series as

$$\left.\begin{aligned} w &= w_0+\lambda w_1+\lambda^2 w_2+\dots, \\ \phi &= \lambda\phi_1+\lambda^2\phi_2+\dots. \end{aligned}\right\} \tag{2.2}$$

The zero-velocity condition on the periphery may be expressed as

$$w_i = 0, \quad \phi_i = 0, \quad \frac{d\phi}{d\xi} = 0, \tag{2.3}$$

where $d/d\xi$ represents differentiation in the normal direction to the boundary in the meridian plane.

Substitution of w and ϕ from (2.2) into the first equation of (1.8) and comparison of the coefficients of λ^0 yield

$$\nabla^2 w_0 = -k_0. \tag{2.4}$$

The solution of this with the boundary condition $w_0 = 0$ on the periphery is

$$\left.\begin{aligned} w_0 &= Re\, c\left(1-\frac{y^2}{a_1^2}-\frac{x^2}{b_1^2}\right) \quad \text{for a horizontally placed ellipse,} \\ w_0 &= Re\, c\left(1-\frac{y^2}{b_1^2}-\frac{x^2}{a_1^2}\right) \quad \text{for a vertically placed ellipse,} \end{aligned}\right\} \tag{2.5}$$

where a_1 and b_1 are the dimensionless semimajor and semiminor axes defined relative to the unit length A, and

$$c = \frac{a_1^2 b_1^2}{1+m^4}.$$

At this stage it is convenient to introduce the elliptic coordinates (ξ, η) in the cross-sectional plane by the relations

$$\left.\begin{aligned} y &= \frac{1}{1+m^2}\left(\xi+\frac{m^2}{\xi}\right)\cos\eta, \quad x = \frac{1}{1+m^2}\left(\xi-\frac{m^2}{\xi}\right)\sin\eta \\ &\qquad\qquad \text{for a horizontally placed ellipse,} \\ y &= \frac{1}{1-m^2}\left(\xi-\frac{m^2}{\xi}\right)\sin\eta, \quad x = \frac{1}{1-m^2}\left(\xi+\frac{m^2}{\xi}\right)\cos\eta \\ &\qquad\qquad \text{for a vertically placed ellipse,} \end{aligned}\right\} \tag{2.6}$$

for which the ranges of each coordinate are

$$m \leqslant \xi \leqslant 1, \quad 0 \leqslant \eta \leqslant 2\pi.$$

3. Secondary flow for horizontally placed elliptical periphery

Substitution of w and ϕ from (2.2) into the second equation of (1.6) and comparison of the coefficients of λ yield

$$\nabla^4\phi_1 = 2w_0\frac{\partial w_0}{\partial x}. \tag{3.1}$$

Letting $$\phi_1 = -Re^2\left(\frac{1+m^2}{1+m^4}\right)^2 \bar{\phi}_1$$

and using (2.5) and (2.6), (3.1) reduces to

$$\nabla^4\bar{\phi}_1 = 4b(1+m^4-x_2)\,[(1+m^2+m^4)\sin\eta - m^2\sin 3\eta], \tag{3.2}$$

where $$b = \xi - \frac{m^2}{\xi}, \quad x_2 = \xi^2 + \frac{m^4}{\xi^2}.$$

The transformation of the Laplace operator from rectangular coordinates to elliptic coordinates is

$$\nabla^2 = \frac{1}{\xi^2 j^2}\bar{\nabla}^2, \tag{3.3}$$

where $$\bar{\nabla}^2 = \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial \eta^2}, \quad t = \ln\xi,$$

and $$\xi^2 j^2 = x_2 - 2m^2\cos 2\eta.$$

Letting $$\bar{\phi}_1 = \frac{1}{b_1}\bar{\bar{\phi}}_1, \quad b_1 = 1-m^2,$$

and using (3.3), the equation for $\nabla^4\bar{\phi}_1$ transforms into a form involving the operator $\bar{\nabla}^2$ as

$$\bar{\nabla}^2\nabla^2\bar{\bar{\phi}}_1 = b_1 b(u_1\sin\eta + u_3\sin 3\eta + u_5\sin 5\eta), \tag{3.4}$$

where
$$b_1 bu_1 = 4[(1-m^{10})\,e_6 - (1-m^6)\,e_{10}\,\xi^2]\,\xi^3,$$
$$b_1 bu_3 = -4m^2[(1-m^{10})\,e_2 - (1-m^2)\,e_{10}\,\xi^4]\,\xi,$$
$$b_1 bu_5 = 4m^4[(1-m^6)\,e_2 - (1-m^2)\,e_6\,\xi^2]\,\xi$$

and $$e_2 = 1-\frac{m^2}{\xi^2}, \quad e_6 = 1-\frac{m^6}{\xi^6}, \quad e_{10} = 1-\frac{m^{10}}{\xi^{10}}.$$

The form of (3.4) indicates that the solution for $\nabla^2\bar{\bar{\phi}}_1$ must have the form

$$\nabla^2\bar{\bar{\phi}}_1 = L_1\sin\eta + L_3\sin 3\eta + L_5\sin 5\eta, \tag{3.5}$$

where L_1, L_3 and L_5 are functions of ξ.

After taking the Laplacian $\bar{\nabla}^2$ of the right-hand side of (3.5) and comparing the resulting form with the right-hand side of (3.4), one finds the following ordinary differential equations for L_1, L_3 and L_5:

$$\left.\begin{aligned}
\left[\frac{1}{z}(zL_1)'\right]' &= \frac{1}{\xi^2}b_1 bu_1, \quad \text{where } z = \xi,\\
\left[\frac{1}{z}(zL_3)'\right]' &= \frac{1}{9}\frac{1}{\xi^6}b_1 bu_3, \quad \text{where } z = \xi^3,\\
\left[\frac{1}{z}(zL_5)'\right]' &= \frac{1}{25}\frac{1}{\xi^{10}}b_1 bu_5, \quad \text{where } z = \xi^5.
\end{aligned}\right\} \tag{3.6}$$

In each of the above differential equations, primes indicate differentiation with respect to the proper variable z.

After the substitution of u_1, u_3 and u_5 into (3.6), successive integrations yield

$$\left.\begin{aligned} L_1 &= M_1\xi+N_1\frac{1}{\xi}+\frac{1}{6}\left[3(1-m^{10})\left(1-\frac{m^6}{\xi^6}\right)-(1-m^6)\left(1-\frac{m^{10}}{\xi^{10}}\right)\xi^2\right]\xi^3,\\ L_3 &= M_3\xi^3+N_3\frac{1}{\xi^3}+\tfrac{1}{4}m^2\left[2(1-m^{10})\left(1-\frac{m^2}{\xi^2}\right)+(1-m^2)\left(1-\frac{m^{10}}{\xi^{10}}\right)\xi^4\right]\xi,\\ L_5 &= M_5\xi^5+N_5\frac{1}{\xi^5}-\tfrac{1}{12}m^4\left[2(1-m^6)\left(1-\frac{m^2}{\xi^2}\right)-3(1-m^2)\left(1-\frac{m^6}{\xi^6}\right)\xi^2\right]\xi,\end{aligned}\right\}\quad(3.7)$$

where the constants in the first two terms of each expression are integral constants.

Before integrating again, (3.5) must be transformed, by use of (3.3), into a form containing $\overline{\nabla}^2$ as

$$\overline{\nabla}^2\overline{\overline{\phi}}_1 = \bar{u}_1\sin\eta+\bar{u}_3\sin 3\eta+\bar{u}_5\sin 5\eta+\bar{u}_7\sin 7\eta, \quad (3.8)$$

where

$$\left.\begin{aligned} \bar{u}_1 &= (M_1-m^2M_3)\xi^3-m^2(N_3-m^2N_1)\frac{1}{\xi^3}+(N_1+m^2M_1)\left(\xi+\frac{m^2}{\xi}\right)\\ &\quad-\frac{1}{12}\left[2c_6\left(1-\frac{m^6}{\xi^6}\right)-c_{10}\left(1-\frac{m^{10}}{\xi^{10}}\right)\xi^2+2c_{14}\left(1-\frac{m^{14}}{\xi^{14}}\right)\xi^4\right]\xi^3,\\ \bar{u}_3 &= (M_3-m^2M_5)\xi^5+m^2(m^2N_3-N_5)\frac{1}{\xi^5}+m^2(m^2M_3-M_1)\xi+(N_3-m^2N_1)\frac{1}{\xi}\\ &\quad+\tfrac{1}{12}m^2\left[2c_6\left(1-\frac{m^2}{\xi^2}\right)+2c_{14}\left(1-\frac{m^{10}}{\xi^{10}}\right)\xi^4+3b_1\left(1-\frac{m^{14}}{\xi^{14}}\right)\xi^6\right]\xi,\\ \bar{u}_5 &= M_5\xi^7+m^4N_5\frac{1}{\xi^7}-m^2(M_3-m^2M_5)\xi^3+(N_5-m^2N_3)\frac{1}{\xi^3}\\ &\quad-\tfrac{1}{12}m^4\left[c_{10}\left(1-\frac{m^2}{\xi^2}\right)+2c_{14}\left(1-\frac{m^6}{\xi^6}\right)\xi^2\right]\xi,\\ \bar{u}_7 &= -m^2\left(M_5\xi^5+N_5\frac{1}{\xi^5}\right)+\tfrac{1}{12}m^6\left[2c_{14}\left(1-\frac{m^2}{\xi^2}\right)-3b_1\left(1-\frac{m^6}{\xi^6}\right)\xi^2\right]\xi,\end{aligned}\right\}\quad(3.9)$$

where

$$\begin{aligned} b_1 &= 1-m^2,\\ c_6 &= m^2[m^2(1-m^6)-3(1-m^{10})],\\ c_{10} &= 6(1-m^{10})-2m^2(1-m^6)-3m^4(1-m^2),\\ c_{14} &= 1-m^6.\end{aligned}$$

The form of equation (3.8) suggests that the solution $\overline{\overline{\phi}}_1$ should have the form

$$\overline{\overline{\phi}}_1 = F_1\sin\eta+F_3\sin 3\eta+F_5\sin 5\eta+F_7\sin 7\eta, \quad (3.10)$$

where F_1, F_3, F_5 and F_7 are functions of ξ.

Following similar steps for L_1, L_3 and L_5, similar ordinary differential equations are obtained for F_1, F_3, F_5 and F_7. The boundary conditions needed are obtained by the following considerations:

The secondary-flow dimensionless velocity components in the directions of the coordinates ξ and η are

$$q_\xi = -\frac{\sigma}{\eta}\frac{1}{\xi j}\phi_\eta, \quad q_\eta = \frac{\sigma}{\eta}\frac{1}{j}\phi_\xi. \quad (3.11)$$

Using these, the zero-velocity condition on the periphery requires that

$$F_i = 0, \quad \frac{\mathrm{d}F_i}{\mathrm{d}\xi} = 0 \quad \text{when } \xi = 1.$$

An additional condition must be enforced at the points on the line between the two foci, i.e.

$$F_i = 0 \quad \text{when } \xi = m.$$

Employing these boundary conditions, the coefficients of $\overline{\overline{\phi}}_1$ are obtained as

$$F_1 = \frac{1}{288}\left[C_1\left(1-\frac{m^2}{\xi^2}\right) - D_1\left(1-\frac{m^6}{\xi^6}\right)\xi^2 + c_{10}\left(1-\frac{m^{10}}{\xi^{10}}\right)\xi^4 - c_{14}\left(1-\frac{m^{14}}{\xi^{14}}\right)\xi^6\right]\xi, \tag{3.12}$$

where

$$\left.\begin{aligned}
C_1 &= c_{10}\frac{5(1-m^6)(1+m^{10})-3(1+m^6)(1-m^{10})}{3(1-m^2)(1+m^6)-(1+m^2)(1-m^6)} \\
&\qquad - c_{14}\frac{7(1-m^6)(1+m^{14})-3(1+m^6)(1-m^{14})}{3(1-m^2)(1+m^6)-(1+m^2)(1-m^6)}, \\
D_1 &= c_{10}\frac{5(1-m^2)(1+m^{10})-(1+m^2)(1-m^{10})}{3(1-m^2)(1+m^6)-(1+m^2)(1-m^6)} \\
&\qquad - c_{14}\frac{7(1-m^2)(1+m^{14})-(1+m^2)(1-m^{14})}{3(1-m^2)(1+m^6)-(1+m^2)(1-m^6)}, \\
F_3 &= -\tfrac{1}{480}m^2\left[\tfrac{5}{3}D_1\left(1-\frac{m^2}{\xi^2}\right) - C_3\left(1-\frac{m^6}{\xi^6}\right)\xi^2 + D_3\left(1-\frac{m^{10}}{\xi^{10}}\right)\xi^4 - 3b_1\left(1-\frac{m^{14}}{\xi^{14}}\right)\xi^6\right]\xi,
\end{aligned}\right\} \tag{3.13}$$

$$\left.\begin{aligned}
C_3 &= 3b_1\frac{7(1-m^{10})(1+m^{14})-5(1+m^{10})(1-m^{14})}{5(1-m^6)(1+m^{10})-3(1+m^6)(1-m^{10})} \\
&\qquad + \tfrac{5}{3}D_1\frac{5(1-m^2)(1+m^{10})-(1+m^2)(1-m^{10})}{5(1-m^6)(1+m^{10})-3(1+m^6)(1-m^{10})}, \\
D_3 &= 3b_1\frac{3(1+m^6)(1-m^{14})-7(1-m^6)(1+m^{14})}{3(1+m^6)(1-m^{10})-5(1-m^6)(1+m^{10})} \\
&\qquad - \tfrac{5}{3}D_1\frac{3(1-m^2)(1+m^6)-(1+m^2)(1-m^6)}{3(1+m^6)(1-m^{10})-5(1-m^6)(1+m^{10})}, \\
F_5 &= \tfrac{1}{288}m^4\left[c_{10}\left(1-\frac{m^2}{\xi^2}\right) - \tfrac{3}{5}D_3\left(1-\frac{m^6}{\xi^6}\right)\xi^2 - C_5\left(1-\frac{m^{10}}{\xi^{10}}\right)\xi^4 + D_5\left(1-\frac{m^{14}}{\xi^{14}}\right)\xi^6\right]\xi,
\end{aligned}\right\} \tag{3.14}$$

$$\left.\begin{aligned}
C_5 &= c_{10}\frac{7(1-m^2)(1+m^{14})-(1+m^2)(1-m^{14})}{7(1-m^{10})(1+m^{14})-5(1+m^{10})(1-m^{14})} \\
&\qquad - \tfrac{3}{5}D_3\frac{7(1-m^6)(1+m^{14})-3(1+m^6)(1-m^{14})}{7(1-m^{10})(1+m^{14})-5(1+m^{10})(1-m^{14})}, \\
D_5 &= c_{10}\frac{5(1-m^2)(1+m^{14})-(1+m^2)(1-m^{10})}{7(1-m^{10})(1+m^{14})-5(1+m^{10})(1-m^{14})} \\
&\qquad - \tfrac{3}{5}D_3\frac{5(1-m^6)(1+m^{10})-3(1+m^6)(1-m^{10})}{7(1-m^{10})(1+m^{14})-5(1+m^{10})(1-m^{14})}, \\
F_7 &= -\tfrac{1}{1440}m^6\left[5c_{14}\left(1-\frac{m^2}{\xi^2}\right) - 9b_1\left(1-\frac{m^6}{\xi^6}\right)\xi^2 + C_7\left(1-\frac{m^{10}}{\xi^{10}}\right)\xi^4 - D_7\left(1-\frac{m^{14}}{\xi^{14}}\right)\xi^6\right]\xi,
\end{aligned}\right\} \tag{3.15}$$

$$\left.\begin{aligned} C_7 &= 9b_1 \frac{5(1-m^6)(1+m^{14})-3(1+m^6)(1-m^{14})}{7(1-m^{10})(1+m^{14})-5(1+m^{10})(1-m^{14})} \\ &\qquad -5c_{14}\frac{7(1-m^2)(1+m^{14})-(1+m^2)(1-m^{14})}{7(1-m^{10})(1+m^{14})-5(1+m^{10})(1-m^{14})}, \\ D_7 &= 9b_1 \frac{5(1-m^6)(1+m^{10})-3(1+m^6)(1-m^{10})}{7(1-m^{10})(1+m^{14})-5(1+m^{10})(1-m^{14})} \\ &\qquad -5c_{14}\frac{5(1-m^2)(1+m^{10})-(1+m^2)(1-m^{10})}{7(1-m^{10})(1+m^{14})-5(1+m^{10})(1-m^{14})}. \end{aligned}\right\} \quad (3.16)$$

It may be noted that the constant D_7 can be related to the constant D_5. However, they are listed above separately to maintain the symmetric presentation.

4. Secondary flow for vertically placed elliptical periphery

For the vertical positioning of the elliptic section it is convenient to define $\bar{\phi}_1$ by the relation

$$\phi_1 = -Re^2\left(\frac{1-m^2}{1+m^4}\right)^2 \bar{\phi}_1. \quad (4.1)$$

Use of (4.1) and substitution of w_0 from (2.5) into (3.1) yield

$$\nabla^4\bar{\phi}_1 = 4a(1+m^4-x_2)[(1-m^2+m^4)\cos\eta - m^2\cos 3\eta], \quad (4.2)$$

where $$a = \xi + \frac{m^2}{\xi}.$$

Using the transformation relation (3.3) and letting

$$\bar{\phi}_1 = \frac{1}{a_1}\bar{\bar{\phi}}_1, \quad a_1 = 1+m^2,$$

the equation of $\bar{\phi}_1$ is transformed into a form involving the operator $\bar{\nabla}^2$ as

$$\bar{\nabla}^2\nabla^2\bar{\bar{\phi}}_1 = a_1 a(u_1\cos\eta + u_3\cos 3\eta + u_5\cos 5\eta), \quad (4.3)$$

where
$$a_1 a u_1 = 4[(1+m^6)e_6 - (1+m^6)e_{10}\xi^2]\xi^3,$$
$$a_1 a u_3 = -4m^2[(1+m^{10})e_2 - (1+m^2)e_{10}\xi^4]\xi,$$
$$a_1 a u_5 = 4m^4[(1+m^6)e_2 - (1+m^2)e_6\xi^2]\xi$$

and
$$e_2 = 1+\frac{m^2}{\xi^2}, \quad e_6 = 1+\frac{m^6}{\xi^6}, \quad e_{10} = 1+\frac{m^{10}}{\xi^{10}}.$$

It must be noted that, to maintain uniformity, the same notation (u_1, u_3, u_5) used in §3 has been used here.

By following similar steps as those used in §3, (4.3) accepts a solution of the form

$$\bar{\bar{\phi}}_1 = F_1\cos\eta + F_3\cos 3\eta + F_5\cos 5\eta + F_7\cos 7\eta. \quad (4.4)$$

The coefficients involved in (4.4) are calculated as follows:

$$F_1 = \frac{1}{288}\left[C_1\left(1+\frac{m^2}{\xi^2}\right)-D_1\left(1+\frac{m^6}{\xi^6}\right)\xi^2+c_{10}\left(1+\frac{m^{10}}{\xi^{10}}\right)\xi^4-c_{14}\left(1+\frac{m^{14}}{\xi^{14}}\right)\xi^6\right]\xi, \quad (4.5)$$

where

$$\left.\begin{aligned}
&c_{10} = 6(1+m^{10})+2m^2(1+m^6)-3m^4(1+m^2), \quad c_{14} = 1+m^6,\\
&C_1 = c_{10}\frac{5(1+m^6)(1-m^{10})-3(1-m^6)(1+m^{10})}{3(1+m^2)(1-m^6)-(1-m^2)(1+m^6)}\\
&\qquad -c_{14}\frac{7(1+m^6)(1-m^{14})-3(1-m^6)(1+m^{14})}{3(1+m^2)(1-m^6)-(1-m^2)(1+m^6)},\\
&D_1 = c_{10}\frac{5(1+m^2)(1-m^{10})-(1-m^2)(1+m^{10})}{3(1+m^2)(1-m^6)-(1-m^2)(1+m^6)}\\
&\qquad -c_{14}\frac{7(1+m^2)(1-m^{14})-(1-m^2)(1+m^{14})}{3(1+m^2)(1-m^6)-(1-m^2)(1+m^6)},\\
&F_3 = -\tfrac{1}{480}m^2\left[\tfrac{5}{3}D_1\left(1+\frac{m^2}{\xi^2}\right)-C_3\left(1+\frac{m^6}{\xi^6}\right)\xi^2+D_3\left(1+\frac{m^{10}}{\xi^{10}}\right)\xi^4-3a_1\left(1+\frac{m^{14}}{\xi^{14}}\right)\xi^6\right]\xi,
\end{aligned}\right\} \quad (4.6)$$

$$\left.\begin{aligned}
&a_1 = 1+m^2,\\
&C_3 = 3a_1\frac{7(1+m^{10})(1-m^{14})-5(1-m^{10})(1+m^{14})}{5(1+m^6)(1-m^{10})-3(1-m^6)(1+m^{10})}\\
&\qquad +\tfrac{5}{3}D_1\frac{5(1+m^2)(1-m^{10})-(1-m^2)(1+m^{10})}{5(1+m^6)(1-m^{10})-3(1-m^6)(1+m^{10})},\\
&D_3 = 3a_1\frac{3(1-m^6)(1+m^{14})-7(1+m^6)(1-m^{14})}{3(1-m^2)(1+m^{10})-5(1+m^6)(1-m^{10})}\\
&\qquad -\tfrac{5}{3}D_1\frac{3(1+m^2)(1-m^6)-(1-m^2)(1+m^6)}{3(1-m^2)(1+m^{10})-5(1+m^6)(1-m^{10})},\\
&F_5 = \tfrac{1}{288}m^4\left[c_{10}\left(1+\frac{m^2}{\xi^2}\right)-\tfrac{3}{5}D_3\left(1+\frac{m^6}{\xi^6}\right)\xi^2-C_5\left(1+\frac{m^{10}}{\xi^{10}}\right)\xi^4+D_5\left(1+\frac{m^{14}}{\xi^{14}}\right)\xi^6\right]\xi,
\end{aligned}\right\} \quad (4.7)$$

$$\left.\begin{aligned}
&C_5 = c_{10}\frac{7(1+m^2)(1-m^{14})-(1-m^2)(1+m^{14})}{7(1+m^{10})(1-m^{14})-5(1-m^{10})(1+m^{14})}\\
&\qquad -\tfrac{3}{5}D_3\frac{7(1+m^6)(1-m^{14})-3(1-m^6)(1+m^{14})}{7(1+m^{10})(1-m^{14})-5(1-m^{10})(1+m^{14})},\\
&D_5 = c_{10}\frac{5(1+m^2)(1-m^{10})-(1-m^2)(1+m^{10})}{7(1+m^{10})(1-m^{14})-5(1-m^{10})(1+m^{14})}\\
&\qquad -\tfrac{3}{5}D_3\frac{5(1+m^6)(1-m^{10})-3(1-m^6)(1+m^{10})}{7(1+m^{10})(1-m^{14})-5(1-m^{10})(1+m^{14})},\\
&F_7 = -\tfrac{1}{1440}m^6\left[5c_{14}\left(1+\frac{m^2}{\xi^2}\right)-9a_1\left(1+\frac{m^6}{\xi^6}\right)\xi^2+C_7\left(1+\frac{m^{10}}{\xi^{10}}\right)\xi^4-D_7\left(1+\frac{m^{14}}{\xi^{14}}\right)\xi^6\right]\xi,
\end{aligned}\right\} \quad (4.8)$$

$$\left.\begin{aligned}
C_7 &= 9a_1\,\frac{7(1+m^6)(1-m^{14})-3(1-m^6)(1+m^{14})}{7(1+m^{10})(1-m^{14})-5(1-m^{10})(1+m^{14})} \\
&\qquad -5c_{14}\,\frac{7(1+m^2)(1-m^{14})-(1-m^2)(1+m^{14})}{7(1+m^{10})(1-m^{14})-5(1-m^{10})(1+m^{14})}, \\
D_7 &= 9a_1\,\frac{5(1+m^6)(1-m^{10})-3(1-m^6)(1+m^{10})}{7(1+m^{10})(1-m^{14})-5(1-m^{10})(1+m^{14})} \\
&\qquad -5c_{14}\,\frac{5(1+m^2)(1-m^{10})-(1-m^2)(1+m^{10})}{7(1+m^{10})(1-m^{14})-5(1-m^{10})(1+m^{14})}.
\end{aligned}\right\} \tag{4.9}$$

Again the same argument as before holds true that the constants D_7 and D_5 are related to each other.

5. Secondary-flow streamlines and discussion

The secondary-flow streamlines are plotted for a particular section with semiaxis ratio of

$$B_1/A_1 = \tfrac{3}{4},$$

corresponding to an ellipticity ratio of $m = \sqrt{\tfrac{1}{7}}$. The resulting curves are presented in figures 3 and 4. The numerical (negative) values of the stream function $\bar{\bar{\phi}}_1$ are indicated at the eyepoints in each figure.

It is interesting to compare the relative position of the eyepoint for circular and elliptical cross-sections. For a full circular section the eyepoint is located at a point on the vertical axis at a distance of 0.429 of the radius, measured from the centre (cf. Topakoglu 1967).

For an elliptical cross-section with an ellipticity ratio as selected above, the eyepoint positions measured from the centre of the sections are, for a horizontally placed ellipse, 0.422 of the semiminor axis; and, for a vertically placed ellipse, 0.482 of the semimajor axis.

It must be noted that, although the streamlines as calculated from the first term of the expansion of the secondary flow are symmetrical with respect to the vertical axis CX (figure 1), the velocity components from (1.4) are not symmetrical, because of the term Y.

A comparison is made between the results of Thomas & Walters (1965) and the present findings. Noting that the comparison must be based on the dimensional velocity components, and also noting that the perturbation parameter of Thomas & Walters (i.e. the Dean number) when written in the notation used in this paper is

$$2\,\frac{A_1}{\Sigma}\,Re^2,$$

the horizontal velocity components of the present paper and of Thomas & Walters (when written in the notation of this paper) are respectively

$$\left.\begin{aligned}
V &= \frac{\nu}{A}\,\frac{\sigma}{\sigma+y}\,Re^2\,\frac{1+m^2}{(1+m^4)^2}\,\frac{\partial\bar{\bar{\phi}}_1}{\partial x}, \\
V &= -\frac{\nu}{A}\,\frac{1}{\sigma}\,Re^2\,\frac{1+m^2}{(1+m^4)^2}\,2(1+m^4)^2\,\frac{\partial\chi_1}{\partial x}.
\end{aligned}\right\} \tag{5.1}$$

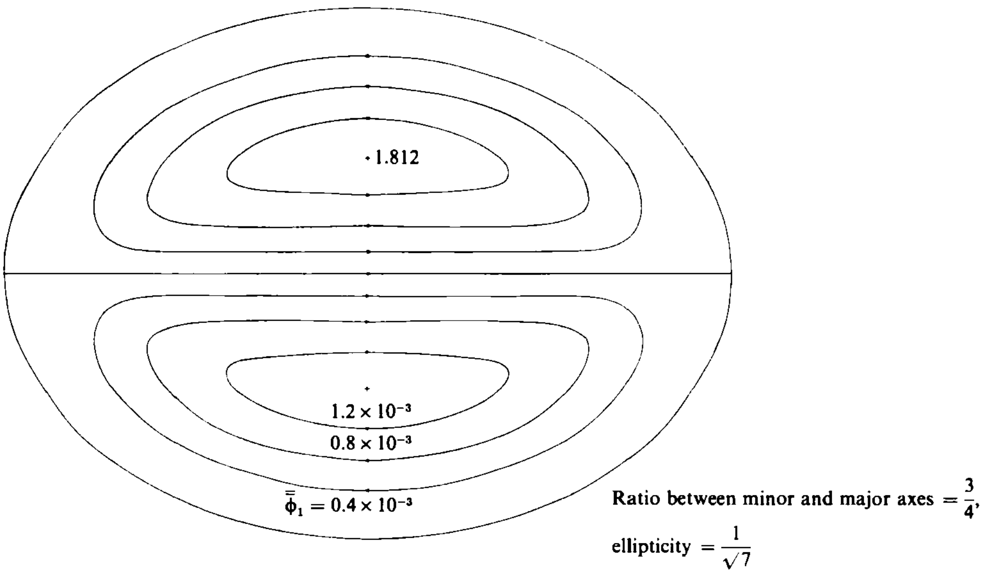


FIGURE 3. Secondary-flow streamlines for horizontally placed ellipse.

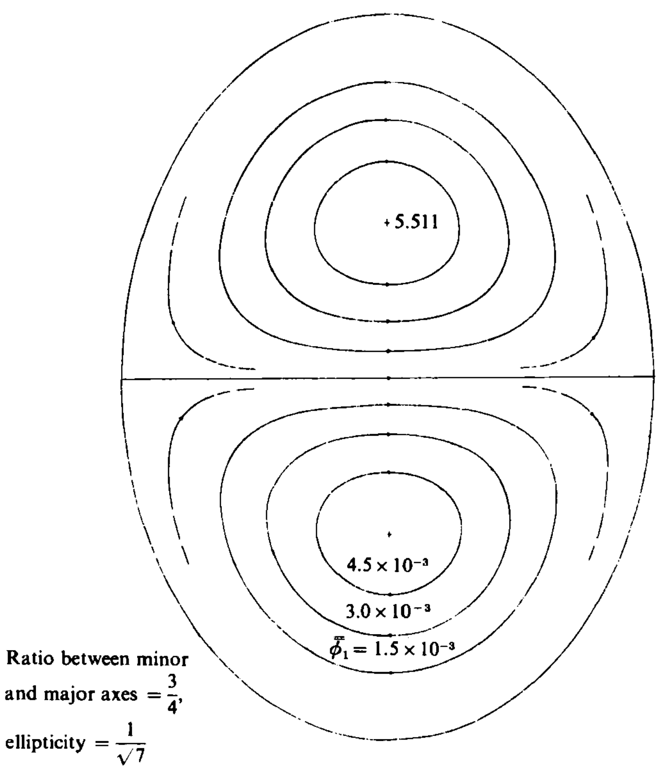


FIGURE 4. Secondary-flow streamlines for vertically placed ellipse.

x	y	$\sigma\overline{\overline{\phi}}_1$	$-2(1+m^4)^2\chi_1$
0	0.444324	0	0
0.034672	0.445037	0.000207	0.000199
0.173361	0.461831	0.000901	0.000869
0.282475	0.489431	0.001134	0.001096
0.292787	0.492620	0.001137	0.001099
0.401206	0.531406	0.000983	0.000953
0.502746	0.575190	0.000597	0.000580
0.599700	0.622306	0.000189	0.000185
0.693443	0.671755	0	0

TABLE 1. Comparison of stream function with that of Thomas & Walters (1965)

In Thomas & Walters' work a left-handed coordinate system is used. Therefore the first V of (5.1) must be compared with the second V of (5.1) after changing its sign. In addition, since Thomas & Walters' analysis is based on the simplified continuity and simplified Navier–Stokes equations, their velocity components do not contain the term Y as seen in (1.4). Another difference between the two analyses is the use of ϕ, as defined in (1.5), which is not used in Thomas & Walters' formulation. The above considerations require that the function $\sigma\overline{\overline{\phi}}_1$ of the present paper must be compared with $-2(1+m^4)^2\chi_1$ of Thomas & Walters' result at the corresponding points of the cross-section. The numerical values of these two quantities calculated at some selected points on the line, $\eta = 0.6\pi/2$, for a horizontally placed ellipse with an ellipticity ratio $m = \sqrt{\frac{1}{7}}$, are shown in table 1.

The table shows that there are still differences between the two sets of results. Some of these are unavoidable because of long numerical computations. Further analytical inspection is impossible because details have not been given for the integration of the non-homogeneous biharmonic equation of the secondary-flow stream function in Thomas & Walters' work.

In conclusion, it must be added that, in addition to the correct effect of the factor Y in (1.4), the availability of expressions in elliptic coordinates will facilitate further calculations for higher-order terms of the primary and secondary flows in elliptic curved pipes as well as in other problems involving flow in such pipes.

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